Effect of coordinate noncommutativity on the mass of a particle in a uniform field and the equivalence principle

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Abstract

We consider the motion of a particle in a uniform field in noncommutative space which is rotationally invariant. On the basis of exact calculations it is shown that there is an effect of coordinate noncommutativity on the mass of a particle. A particular case of motion of a particle in a uniform gravitational field is considered and the equivalence principle is studied. We propose the way to solve the problem of violation of the equivalence principle in the rotationally invariant noncommutative space.

Key words: noncommutative space; rotational symmetry; equivalence principle.

1   Introduction

The idea of noncommutative structure of space was suggested by Heisenberg and later formalized by Snyder [1]. In recent years, due to the development of String Theory and Quantum Gravity noncommutativity has received a considerable interest (see, for instance, [2, 3]).

In the canonical version of noncommutative space the coordinate and the momentum operators satisfy the following commutation relations

$$[X_i, X_j] = i\hbar\theta_{ij}, \tag{1}$$

$$[X_i, P_j] = i\hbar\delta_{ij}, \tag{2}$$

$$[P_i, P_j] = 0, \tag{3}$$

with θ_{ij} being a constant antisymmetric matrix. Much attention has been devoted to studies of different physical problems in such a space, among them the hydrogen atom (see, for example, [4, 5, 6, 7, 8, 9, 10, 11]), the Landau problem (see, for example, [12, 13, 14, 15, 16]), quantum mechanical system in a central potential [17], classical particle in a gravitational potential [18, 19], system of particles in a gravitational field [20], motion of a body in a gravitational field and the equivalence principle [21], a particle in a gravitational quantum well in the case of canonical noncommutativity of coordinates and momentum-momentum noncommutativity [22, 23], and many others.

It is worth noting that in the case of canonical version of noncommutative space (1)-(3) one faces the problem of rotational symmetry breaking [4, 24]. Therefore, different classes

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of noncommutative algebras were considered to preserve the rotational symmetry (see, for instance, [25, 26] and references therein).

It this paper we consider the motion of a particle in a uniform field in rotationally invariant noncommutative space which was proposed in our previous paper [25]. We show that there is an effect of noncommutativity on the mass of a particle. A particular case of motion of a particle in a uniform gravitational field is considered and the equivalence principle is studied. We propose the way to solve an important problem of violation of the equivalence principle in rotationally invariant noncommutative space and find the condition to recover this principle.

The paper is organized as follows. In Section 2, we consider a noncommutative space with preserved rotational symmetry. In Section 3, the motion of a particle in a uniform field in noncommutative space (6)-(8) is studied. It is shown that there is an effect of noncommutativity on the mass of a particle. In Section 4, we consider the motion of a particle in a uniform gravitational field and study the equivalence principle. Conclusions are presented in Section 5.

2   Noncommutative space with preserved rotational symmetry

In previous paper [25] we studied the problem of rotational symmetry breaking in noncommutative space. To preserve this symmetry we considered a generalization of constant antisymmetric matrix θ_{ij} to a tensor constructed with the help of additional coordinates which are governed by a rotationally symmetric system. We supposed, for simplicity, that these coordinates are governed by the harmonic oscillator. According to the suggestion presented in [25] in the present paper we consider the tensor of noncommutativity which is constructed in the following form

$$\theta_{ij} = \frac{l_0}{\hbar}\varepsilon_{ijk}a_k, \tag{4}$$

where l_0 is a constant with the dimension of length, a_i are additional coordinates governed by the harmonic oscillator with parameters m_{osc} and ω

$$H_{osc} = \frac{(p^a)^2}{2m_{osc}} + \frac{m_{osc}\omega^2 a^2}{2}. \tag{5}$$

We consider the frequency of harmonic oscillator to be very large. In this case the distance between the energy levels of harmonic oscillator is very large too. Therefore, harmonic oscillator put into the ground state remains in it.

So, we consider the following rotationally invariant noncommutative algebra

$$[X_i, X_j] = i\varepsilon_{ijk}l_0 a_k, \tag{6}$$

$$[X_i, P_j] = i\hbar\delta_{ij}, \tag{7}$$

$$[P_i, P_j] = 0. \tag{8}$$

The coordinates a_i and momenta p_i^a satisfy the ordinary commutation relations $[a_i, a_j] = 0$, $[a_i, p_j^a] = i\hbar\delta_{ij}$, $[p_i^a, p_j^a] = 0$. Also, a_i commute with X_i and P_i. Therefore, tensor of noncommutativity (4) commutes with X_i and P_i too. So, X_i, P_i and θ_{ij} satisfy the same commutation relations as in the case of the canonical version of noncommutativity, moreover algebra (6)-(8) is rotationally invariant.

We would like to mention that additional coordinates a_i which form the tensor of noncommutativity can be treated as some internal coordinates of a particle. Quantum fluctuations of these coordinates lead effectively to a non-point-like particle, size of which is of the order of the Planck scale.

Coordinates X_i and momenta P_i can be represented by the coordinates x_i and momenta p_i which satisfy the ordinary commutation relations

$$X_i = x_i - \frac{1}{2}\theta_{ij}p_j, \tag{9}$$
$$P_i = p_i, \tag{10}$$

where θ_{ij} is given by (4). Coordinates x_i and momenta p_i satisfy the following relations

$$[x_i, x_j] = 0, \tag{11}$$
$$[p_i, p_j] = 0, \tag{12}$$
$$[x_i, p_j] = i\hbar\delta_{ij}, \tag{13}$$

and commute with a_i, p_i^a, namely $[x_i, a_j] = 0$, $[x_i, p_j^a] = 0$, $[p_i, a_j] = 0$, $[p_i, p_j^a] = 0$. It is worth mentioning that coordinates X_i do not commute with p_j^a. Taking into account (4) and (9), we have $[X_i, p_j^a] = i\varepsilon_{ijk}l_0 p_k/2$.

Explicit representation for coordinates X_i (9) and momenta P_i (10) guarantee that the Jacobi identity is satisfied. This can be easily checked for all possible triplets of operators.

Algebra (6)-(8) is manifestly rotationally invariant. It is clear that commutation relation (6) remains the same after rotation $X_i' = U(\varphi)X_iU^+(\varphi)$, $a_i' = U(\varphi)a_iU^+(\varphi)$. We have

$$[X_i', X_j'] = i\varepsilon_{ijk}l_0 a_k', \tag{14}$$

where the rotation operator reads $U(\varphi) = e^{\frac{i}{\hbar}\varphi(\mathbf{n}\cdot\tilde{\mathbf{L}})}$. Here $\tilde{\mathbf{L}}$ is the total angular momentum which we define as follows

$$\tilde{\mathbf{L}} = [\mathbf{x} \times \mathbf{p}] + [\mathbf{a} \times \mathbf{p}^a], \tag{15}$$

or taking into account (4), (9) and (10), we have

$$\tilde{\mathbf{L}} = [\mathbf{X} \times \mathbf{P}] + \frac{l_0}{2\hbar}[\mathbf{P} \times [\mathbf{a} \times \mathbf{P}]] + [\mathbf{a} \times \mathbf{p}^a], \tag{16}$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{X} = (X_1, X_2, X_3)$ and $\mathbf{a} = (a_1, a_2, a_3)$. It is worth mentioning that $\tilde{\mathbf{L}}$ satisfies commutation relations which are the same as in ordinary space, namely $[X_i, \tilde{L}_j] = i\hbar\varepsilon_{ijk}X_k$, $[P_i, \tilde{L}_j] = i\hbar\varepsilon_{ijk}P_k$, $[a_i, \tilde{L}_j] = i\hbar\varepsilon_{ijk}a_k$, $[p_i^a, \tilde{L}_j] = i\hbar\varepsilon_{ijk}p_k^a$.

At the end of this Section we would like to note that taking into account (9), we have that the operators X_i depend on the momenta p_i and therefore depend on the mass m. It is clear that operators X_i do not depend on the mass in the case when θ_{ij} is proportional to $1/m$. So, in this case X_i can be considered as a kinematic variables. This condition will be studied in details in Section 4.

3 Motion of a particle in a uniform field in noncommutative space with preserved rotational symmetry

In this section we study the motion of a particle of mass m in a uniform field in rotationally invariant noncommutative space (6)-(8). Let us consider the case when the field is pointed in the X_3 direction. So, the Hamiltonian of the particle is as follows

$$H_p = \frac{P^2}{2m} + \kappa X_3, \tag{17}$$

where the uniform field is characterized by the factor κ. For example, in a particular case of motion of a charged particle q in the uniform electric field E directed along the X_3 axis, factor κ reads $\kappa = -qE$. In the case of motion of a particle of mass m in the uniform gravitational field g directed along the X_3 axis we have $\kappa = -mg$.

In rotationally invariant noncommutative space (6)-(8), because of definition of the tensor of noncommutativity (4), we have to take into account the additional terms which correspond to the harmonic oscillator (5). So, we consider the total Hamiltonian as follows

$$H = H_p + H_{osc} = \frac{P^2}{2m} + \kappa X_3 + \frac{(p^a)^2}{2m_{osc}} + \frac{m_{osc}\omega^2 a^2}{2}. \tag{18}$$

Using representation (9), (10), we can rewrite Hamiltonian (18) in the following form

$$H = \frac{p^2}{2m} + \kappa x_3 + \frac{\kappa l_0}{2\hbar}(a_1 p_2 - a_2 p_1) + \frac{(p^a)^2}{2m_{osc}} + \frac{m_{osc}\omega^2 a^2}{2}. \tag{19}$$

After algebraic transformations, we obtain

$$H = \left(1 - \frac{\kappa^2 l_0^2 m}{4\hbar^2 \omega^2 m_{osc}}\right)\frac{p_1^2}{2m} + \left(1 - \frac{\kappa^2 l_0^2 m}{4\hbar^2 \omega^2 m_{osc}}\right)\frac{p_2^2}{2m} + \frac{p_3^2}{2m} + \kappa x_3 + \frac{(p^a)^2}{2m_{osc}}$$
$$+ \frac{m_{osc}\omega^2}{2}\left(a_1 + \frac{\kappa l_0}{2\hbar\omega^2 m_{osc}}p_2\right)^2 + \frac{m_{osc}\omega^2}{2}\left(a_2 - \frac{\kappa l_0}{2\hbar\omega^2 m_{osc}}p_1\right)^2 + \frac{m_{osc}\omega^2 a_3^2}{2}. \tag{20}$$

Hamiltonian (20) can be rewritten as follows

$$H = \tilde{H}_p + \tilde{H}_{osc}, \tag{21}$$

where

$$\tilde{H}_p = \frac{p_1^2}{2m_{eff}} + \frac{p_2^2}{2m_{eff}} + \frac{p_3^2}{2m} + \kappa x_3, \tag{22}$$

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with m_{eff} being an effective mass which is defined as

$$m_{eff} = m \left(1 - \frac{\kappa^2 l_0^2 m}{4\hbar^2 \omega^2 m_{osc}} \right)^{-1}, \tag{23}$$

and

$$\tilde{H}_{osc} = \frac{(p^a)^2}{2m_{osc}} + \frac{m_{osc}\omega^2 q^2}{2}. \tag{24}$$

Here the components of $\mathbf{q}$ are the following

$$q_1 = a_1 + \frac{\kappa l_0}{2\hbar\omega^2 m_{osc}} p_2, \tag{25}$$

$$q_2 = a_2 - \frac{\kappa l_0}{2\hbar\omega^2 m_{osc}} p_1, \tag{26}$$

$$q_3 = a_3. \tag{27}$$

Note that q_i satisfy the following commutation relations $[q_i, q_j] = 0$, $[q_i, p_j^a] = i\hbar\delta_{ij}$, also $[q_i, x_j] = -i\varepsilon_{ij3}\kappa l_0/(2m_{osc}\omega^2)$, $[q_i, p_j] = 0$. So, Hamiltonian $\tilde{H}_{osc}$ corresponds to the tree-dimensional harmonic oscillator in the ordinary space.

It is important that $p_1^2/(2m_{eff}) = \tilde{H}_1$, $p_2^2/(2m_{eff}) = \tilde{H}_2$, $p_3^2/(2m) + \kappa x_3 = \tilde{H}_3$ and $\tilde{H}_{osc}$ commute with each other. The eigenfunctions of $H = \tilde{H}_1 + \tilde{H}_2 + \tilde{H}_3 + \tilde{H}_{osc}$ (21) can be written as follows

$$\psi(\mathbf{x}, \tilde{\mathbf{q}}) = Ce^{ik_1 x_1} e^{ik_2 x_2} \psi^{(3)}(x_3) \psi^{\tilde{q}}(\tilde{\mathbf{q}}), \tag{28}$$

here C is a constant, k_1 and k_2 are the components of the wave vector corresponding to the free motion of a particle in the perpendicular directions to the field direction, $\psi^{(3)}(x_3)$ are well known eigenfunctions of $\tilde{H}_3$ which correspond to the motion of a particle in the field direction and can be written in terms of the Airy function, and $\psi^{\tilde{q}}(\tilde{\mathbf{q}})$ are eigenfunctions of tree-dimensional harmonic oscillator with parameters m_{osc} and ω. The components of $\tilde{\mathbf{q}}$ read $\tilde{q}_1 = a_1 + \kappa l_0 k_2/(2\omega^2 m_{osc})$, $\tilde{q}_2 = a_2 - \kappa l_0 k_1/(2\omega^2 m_{osc})$ and $\tilde{q}_3 = a_3$.

Let us write the eigenvalues of H (21). Taking into account that the harmonic oscillator is in the ground state, we have

$$E = \frac{\hbar^2 k_1^2}{2m_{eff}} + \frac{\hbar^2 k_2^2}{2m_{eff}} + E_3 + \frac{1}{2}\hbar\omega, \tag{29}$$

where E_3 corresponds to the motion of a particle in the field direction.

Taking into account first two terms in (29), we can conclude that there is an effect of coordinate noncommutativity on the mass of a particle. It is worth mentioning that noncommutativity has an effect on the motion of a particle in perpendicular directions to the direction of uniform field. The motion of a particle in the field direction is governed by $\tilde{H}_3$ and is the same as in the ordinary space. Therefore, noncommutativity of coordinates causes the anisotropy of mass.

At the end of this section we would like to note that because of the rotationally invariance obtained in this Section results can be easy generalized to the case of an arbitrary direction of the uniform field.

In the next section, using obtained results, we consider a particular case of motion of a particle in a uniform gravitational field and study the equivalence principle.

4 Motion of a particle in a uniform gravitational field. Equivalence principle

In this section we study the free fall of a particle in the uniform gravitational field in noncommutative space (6)-(8). The gravitational field is directed along the X_3 axis and is characterized by the factor g. So, on the basis of the results obtained in Section 3, taking into account that $\kappa = -mg$, from (21), (22) and (23) we have

$$H = \frac{p_1^2}{2m_{eff}} + \frac{p_2^2}{2m_{eff}} + \frac{p_3^2}{2m} - mgx_3 + \tilde{H}_{osc}, \tag{30}$$

with

$$m_{eff} = m\left(1 - \frac{l_0^2 g^2 m^3}{4\hbar^2 \omega^2 m_{osc}}\right)^{-1}. \tag{31}$$

Taking into account (31) and (29), we can conclude that because of the term proportional to m^3 in (31) the weak equivalence principle, also known as the universality of free fall or the Galilean equivalence principle, is violated in noncommutative space (6)-(8).

In previous paper [21] we studied the problem of violation of the equivalence principle in two-dimensional noncommutative space in the case of canonical version of noncommutativity $[X_1, X_2] = i\hbar\theta$ with θ being a constant. We proposed the way to solve this problem. It was shown that the equivalence principle is recovered in the case when the following condition is satisfied

$$\theta = \frac{\gamma}{m}, \tag{32}$$

where θ is the parameter of noncommutativity, which corresponds to the particle of mass m, and γ is a constant which takes the same value for all particles.

The problem of violation of the equivalence principle was also studied in deformed space with minimal length $[X, P] = i\hbar(1 + \beta P^2)$, where β is the parameter of deformation [27]. It was shown that this problem is solved in the case when the parameter of deformation is completely determined by the mass of a particle in the following way $\beta m^2 = const$.

Let us find the way to solve the problem of violation of the equivalence principle in rotationally invariant noncommutative space (6)-(8). Taking into account (29) and (31), it is clear

that the necessary condition for the recovering of the equivalence principle is proportionality of the effective mass (31) to the mass of a particle. It can be realized when

$$\frac{l_0^2 m^3}{\omega^2 m_{osc}} = A = const, \tag{33}$$

where A is a constant which takes the same value for particles of different masses. It is convenient to introduce the dimensionless constant

$$\tilde{A} = A\frac{\omega_P^2}{l_P^2 m_P^2}, \tag{34}$$

where l_P and m_P are the Planck length and the Planck mass, respectively, and ω_P is defined as $\hbar\omega_P = E_P$ with E_P being the Planck energy. Therefore, the effective mass reads

$$m_{eff} = m\left(1 - \frac{\tilde{A}l_P^2 m_P^2 g^2}{4\hbar^2 \omega_P^2}\right)^{-1}. \tag{35}$$

At the end of Section 2 we have noted that the operators X_i depend on the mass of a particle. This also violates the equivalence principle. So, let us consider in details the condition which makes the operators X_i independent on the mass of a particle (see the end of Section 2). For this purpose it is convenient to rewrite the tensor of noncommutativity in the following form

$$\theta_{ij} = \frac{l_0 l_{osc}}{\hbar}\varepsilon_{ijk}\tilde{a}_k, \tag{36}$$

where

$$l_{osc} = \sqrt{\frac{\hbar}{m_{osc}\omega}}, \tag{37}$$

and $\tilde{a}_k$ are the dimensionless coordinates $\tilde{a}_k = a_k/l_{osc}$. From (9) and (36), it is clear that the operators X_i do not depend on the mass of a particle in the case when

$$\frac{l_0 l_{osc}}{l_P^2} = \tilde{\gamma}\frac{m_P}{m}, \tag{38}$$

where $\tilde{\gamma}$ is a dimensionless constant which is the same for particles of different masses. For this constant we use notation $\tilde{\gamma}$ in order to distinguish it from γ which was used in the canonical version of noncommutative space in [21]. Taking into account (36), it is clear that condition (38) is similar to the condition (32) which gives the possibility to recover the equivalence principle in a two-dimensional space with canonical noncommutativity of coordinates.

So, conditions (33) and (38) give the possibility to recover the equivalence principle in rotationally invariant noncommutative space (6)-(8).

It is worth mentioning that from (33), (38), using (34) and (37), we obtain

$$\omega = \frac{\tilde{\gamma}^2}{\tilde{A}}\omega_P\frac{m}{m_P}. \tag{39}$$

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Therefore, both conditions are satisfied in the case when the frequency of harmonic oscillator is proportional to the mass of a particle. Taking into account (37), (38) and (39), we also have that parameters l_{osc}, l_0 are inverse to $\sqrt{m}$, namely

$$l_{osc} = l_P \sqrt{\frac{\tilde{A} m_P^2}{\tilde{\gamma}^2 m_{osc} m}}, \tag{40}$$

$$l_0 = l_P \sqrt{\frac{\tilde{\gamma}^4 m_{osc}}{\tilde{A} m}}. \tag{41}$$

Let us estimate the value of $\tilde{A}$ and $\tilde{\gamma}$ constants. For this purpose we suppose that for the electron $l_0 = l_P$, $l_{osc} = l_P$, and $\omega = \omega_P$. In this case, taking into account (37), we have $m_{osc} = m_P$. As a consequence, form (33) and (34), we find

$$\tilde{A} = \frac{m_e^3}{m_P^3} = 7.3 \times 10^{-68}, \tag{42}$$

here m_e is the mass of electron. Also, from (38) we obtain

$$\tilde{\gamma} = \frac{m_e}{m_P} = 4.2 \times 10^{-23}. \tag{43}$$

At the end of this section we would like to mention that in the previous papers [25, 28] the limit $\omega \to \infty$ was considered. Note that in this limit effect of noncommutativity on the mass of a particle tends to zero.

In this paper we consider a finite limit for ω. It is worth mentioning that fixing $\omega = \omega_P$, $l_0 = l_P$, $l_{osc} = l_P$ for the electron, from (39), (40), (41) we can calculate the parameters for a particle of mass m_i. We obtain $\omega^{(i)} = \omega_P m_i / m_e$, $l_{osc}^{(i)} = l_P \sqrt{m_e/m_i}$ and $l_0^{(i)} = l_P \sqrt{m_e/m_i}$.

5   Conclusion

In this paper we have considered the motion of a particle in a uniform field in rotationally invariant noncommutative space (6)-(8). All calculations presented here are exact. We have not made an approximation on the basis of general assumption that the parameter of noncommutativity is small, namely the parameter is of the order of squared Planck length. We have considered the suggestion presented in [25] to construct rotationally invariant noncommutative algebra by the generalization of a constant antisymmetric matrix θ_{ij} to a tensor defined by additional coordinates which are governed by harmonic oscillator. It is worth mentioning that noncommutativity effects on the kinetic terms which correspond to the motion of a particle in perpendicular directions to the direction of the uniform field. The motion of a particle in these directions can be described with the help of effective mass. The motion of a particle in the field direction is the same as in the ordinary space. So, we have concluded that there is an effect of noncommutativity (6) on the mass of a particle in a uniform field and noncommutativity causes the anisotropy of mass.

The particular case of motion of a particle in the uniform gravitational field has been considered. We have studied the problem of violation of the equivalence principle and found the conditions (33), (38) to recover this principle in rotationally invariant noncommutative space (6)-(8). It is important to note that these conditions are in the agreement with the condition which gives the possibility to solve the problem of violation of the equivalence principle in a two-dimensional space with canonical noncommutativity of coordinates [21].

Acknowledgements

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